ELEMENTARY PROOF OF AVERAGE DISTANCE OF TWO RANDOM POINTS ON LINE

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Abstract. We define a line with the length L and two points A and B. The two points are placed randomly on the line. We will derive the average distance between A and B for given L with elementary methods.

The average distance is the expectation value

$$E = \sum_{l \in L} p(l) \cdot l \tag{0.1}$$

over all possible distances $0 \leq l \leq L$. In order to approach the problem we divide the length L into n discrete pieces, so that A and B must be places on one of those pieces. By using infinitely many pieces we can calculate the expectation value by

$$E = \lim_{n\to\infty} \sum_{i=1}^{n} p(i) \cdot L\frac{i}{n} = L \cdot \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} p(i) \cdot i. \tag{0.2}$$

Note: We do not need to consider a distance of $l = 0$ in the expectation value sum.

The probability $p(i)$ is given by the number of possible arrangements $Ar(i)$ of A and B with $|A - B| = i$ over the number of possible arrangements n^2 of A and B:

$$p(i) = \frac{Ar(i)}{n^2}. \tag{0.3}$$

It is obviously $A(1) = 2n$, $A(2) = 2n - 2$, ..., $A(n) = 2$, i.e. if the distance increases by 1 piece, the number of possible arrangements decreases by 2 due to the extra size of l. Therefore we have

$$Ar(i) = 2n - 2(i - 1) = 2n + 2 - 2i. \tag{0.4}$$

We put this in the expectation sum and yield

$$E = L \cdot \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{2n + 2 - 2i}{n^2} \cdot i \tag{0.5}$$

$$= L \cdot \lim_{n\to\infty} \frac{2}{n^3} \sum_{i=1}^{n} ni + i - i^2 \tag{0.6}$$

$$= L \cdot \lim_{n\to\infty} \frac{2}{n^3} \left[(n+1) \sum_{i=1}^{n} i - \sum_{i=1}^{n} i^2 \right]. \tag{0.7}$$

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It is well known that

$$\sum_{i=1}^{n} i = \frac{n^2+n}{2} \quad \text{and} \quad \sum_{i=1}^{n} i^2 = \frac{(2n+1)(n+1)n}{6} \tag{0.8}$$

and can easily be proven by induction. We reduce the sums and continue with the rearrangements

$$E = L \cdot \lim_{n\to\infty} \frac{2}{n^3}\left[\frac{1}{2}(n+1)(n^2+n) - \frac{1}{6}(2n+1)(n+1)n\right] \tag{0.9}$$

$$= L \cdot \lim_{n\to\infty}\left[\frac{n^3+2n^2+n}{n^3} - \frac{1}{3}\cdot\frac{2n^3+3n^2+n}{n^3}\right] \tag{0.10}$$

$$= L \cdot \left[1 - \frac{2}{3}\right] \tag{0.11}$$

$$= \frac{1}{3}L. \tag{0.12}$$

This proves that the average distance of two randomly placed points on a line with length L is given by $1/3 \cdot L$ which concludes the derivation.